

## 10.6 Applications of Quadratic Equations

In this section we want to look at the applications that quadratic equations and functions have in the real world.

There are several standard types: problems where the formula is given, falling object problems, problems involving geometric shapes. Just to name a few. There are many other types of application problems that use quadratic equations, however, we will concentrate on these types to simplify the matter.

We must be very careful when solving these problems since sometimes we want the maximum or minimum of the quadratic, and sometimes we simply want to solve or evaluate the quadratic.

For the problems where we want to find the maximum or minimum value, we recall from the last section that these values always occur at the vertex. Therefore, we will need to find the vertex. When dealing with word problems it is generally easier and more efficient to use the  $x = -\frac{b}{2a}$  method to find the vertex.

Now lets see some examples.

### Example 1:

The number of bacteria in a refrigerated food is given by  $N(T) = 20T^2 - 20T + 120$ , for  $-2 \leq T \leq 14$  and where  $T$  is the temperature of the food in Celsius. At what temperature will the number of bacteria be minimal?

Solution:

First we can see that we have a quadratic function given to us. Moreover, the parabola would open up. So that means that the vertex of the parabola represents a minimum value. This means the first thing we need to do is determine the vertex, since we want the minimal number of bacteria.

As we stated, we will use the formula  $x = -\frac{b}{2a}$  to find the vertex since it is much more efficient than the completing the square method for finding the vertex in word problems. Since our function is in terms of  $T$ , the formula we really use is  $T = -\frac{b}{2a}$ . So we get

$$\begin{aligned} T &= -\frac{-20}{2(20)} \\ &= \frac{20}{40} \\ &= \frac{1}{2} \end{aligned}$$

So the  $T$  value of the vertex is  $\frac{1}{2}$ . Now we must look back at the question to see what we really wanted. Since we want the temperature at which the number is minimum and  $T$  is the temperature in Celsius, this is the value we want. Therefore, the minimum number of bacteria are present when the temperature is  $\frac{1}{2}^{\circ}$  Celsius.

Notice in the last example we only wanted the  $x$  value of the vertex. That is, the value for which the vertex occurs. If the question was asking for what the maximum or minimum value was, we would have had to find the  $y$  value of the vertex.

Always remember, the  $x$  value refers to the value at which the function reaches the maximum or minimum, the  $y$  value refers to what the value of that maximum or minimum actually is.

Example 2:

The height,  $h$ , in feet of an object above the ground is given by  $h = -16t^2 + 64t + 190$ ,  $t \geq 0$  where  $t$  is the time in seconds. Find the time it takes the object to strike the ground and find the maximum height of the object.

Solution:

Let's first find the time it takes for the object to hit the ground. Since  $h$  represents the height above the ground, we would like to know at what time  $h = 0$ . So in the equation

$h = -16t^2 + 64t + 190$  we will set  $h = 0$  and solve for the time,  $t$ .

We have

$$h = -16t^2 + 64t + 190$$

$$0 = -16t^2 + 64t + 190$$

So we simply want to solve this quadratic equation. It is easiest to use the quadratic formula in this situation. So we get

$$\begin{aligned} t &= \frac{-64 \pm \sqrt{64^2 - 4(-16)(190)}}{2(-16)} \\ &= \frac{-64 \pm \sqrt{16256}}{-32} \\ &= \frac{-64 \pm 8\sqrt{254}}{-32} \\ &= \frac{8 \pm \sqrt{254}}{4} \\ &\approx 5.98, -1.98 \end{aligned}$$

However, since  $t$  represents time, we must throw out  $-1.98$ . Therefore, it takes 5.98 seconds for the object to strike the ground.

The other part of the question is we want to know that maximum height that the object reaches. Since we can see that the function is clearly a quadratic function which opens down, we know that this maximum must occur at the vertex. So let's find the vertex. Again we use the formula

$t = -\frac{b}{2a}$ . So we have

$$\begin{aligned} t &= -\frac{64}{2(-16)} \\ &= -\frac{64}{-32} \\ &= 2 \end{aligned}$$

So at  $t = 2$  seconds the object reaches its maximum height. However, we wanted to know what that maximum height is. Therefore, we must find the value of the vertex, in this case it will be the value of  $h$  when  $t = 2$ . So we plug this in to get

$$\begin{aligned} h &= -16(2)^2 + 64(2) + 190 \\ &= 254 \end{aligned}$$

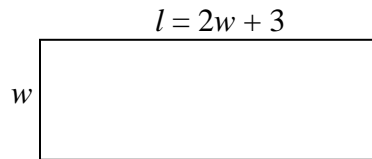
So the maximum height is 254 feet.

Example 3:

The length of a rectangle is three more than twice the width. Determine the dimensions that will give a total area of  $27 \text{ m}^2$ . What is the minimum area that this rectangle can have?

Solution:

First we need to draw a picture to visualize the problem. Since the length is 3 more than twice the width, we will have  $l = 2w + 3$ . So we have the following picture



Now, since both parts of this question deal with the area of this rectangle, let's begin by generating a function for the area. Since  $A = l \cdot w$  we have

$$\begin{aligned} A &= (2w + 3)w \\ &= 2w^2 + 3w \end{aligned}$$

For the first part, we want to know what dimensions make an area of  $27 \text{ m}^2$ . Thus, we can insert 27 for  $A$  into our function and solve for  $w$ . We have

$$\begin{aligned} 27 &= 2w^2 + 3w \\ 2w^2 + 3w - 27 &= 0 \\ (w - 3)(2w + 9) &= 0 \\ w &= 3, -\frac{9}{2} \end{aligned}$$

So, since  $w$  represents the width of a rectangle we must omit the negative value. Therefore, we have  $w = 3$ . Plugging that value into  $l = 2w + 3$  we get  $l = 2 \cdot 3 + 3 = 9$ .

Therefore, the dimensions that give the rectangle an area of  $27 \text{ m}^2$  are 9 m by 3 m.

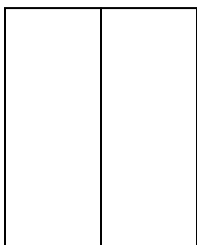
The second part of the problem asks us for the minimum possible area. So going back to our area function,  $A = 2w^2 + 3w$ , we see that the parabola opens up and therefore the minimum would occur at the vertex. So we find the vertex using  $w = -\frac{b}{2a}$  like usual. We get

$$\begin{aligned} w &= -\frac{3}{2 \cdot 2} \\ &= -\frac{3}{4} \end{aligned}$$

But remember,  $w$  represents the width of a rectangle. So, since the minimum occurs when the width is negative, the rectangle must have no minimum area. So we simply say no minimum area exists for this given situation.

Example 4:

Two rectangular corral's are to be made from 100 yds of fencing as seen below.

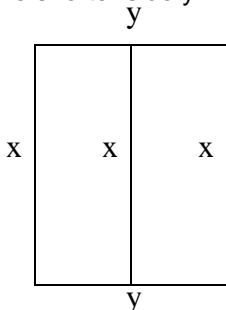


If the rancher wants the total area to be maximum, what dimensions should be used to make the corral's?

Solution:

Since we are looking to maximize the area, we need to generate a function for the area so that we will only need to find the maximum of this area function.

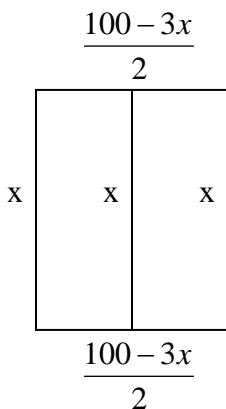
The first thing we should do is decide on some labels and variables. Lets call the longer side  $x$  and the shorter side  $y$ . This gives the following picture:



The first thing we must do is express  $y$  in terms of  $x$ , that way we will only have to deal with one variable. So the total amount of fencing to be used can be represented by  $x+x+x+y+y$ . But we know the total amount to be used is 100yds.

Therefore,  $100=3x+2y$ . So  $y = \frac{100 - 3x}{2}$

Thus we have the picture



So, the total area of the rectangle is

$$A = l \cdot w$$

$$A(x) = x \cdot \frac{100 - 3x}{2}$$

Now that we have a function for the area we need to find the maximum, if one exists.

Notice that we can write our area function as

$$\begin{aligned}
 A(x) &= x \cdot \frac{100 - 3x}{2} \\
 &= x(50 - \frac{3}{2}x) \\
 &= -\frac{3}{2}x^2 + 50x
 \end{aligned}$$

We can see that our function is a quadratic function that opens down. Thus, the vertex will indeed be a maximum. So we proceed in finding the vertex. Again we use the  $x = -\frac{b}{2a}$  approach.

$$\begin{aligned}
 x &= -\frac{b}{2a} \\
 &= -\frac{50}{2(-\frac{3}{2})} \\
 &= -\frac{50}{-3} \\
 &= \frac{50}{3}
 \end{aligned}$$

So the  $x$  value of the vertex is at  $\frac{50}{3}$  or  $16\frac{2}{3}$ . Since  $x$  represented the longer side, we know that the maximum area occurs when longer side has length  $16\frac{2}{3}$  ft. Now looking back at the question, we want the dimensions that will produce the maximum area, not the value of the maximum area. If we were to find  $A(\frac{50}{3})$  we would be finding the value of the maximum area, since  $A$  is the area function. However, recall from the picture we made earlier that  $y = \frac{100 - 3x}{2}$ . Therefore,

$$\begin{aligned}
 y &= \frac{100 - 3(\frac{50}{3})}{2} \\
 &= \frac{100 - 50}{2} \\
 &= 25
 \end{aligned}$$

So the  $y$  value at which the maximum area occurs is 25 ft. So our picture is slightly off since  $y$  is really the longer side. However, we only need to picture to generate our algebraic expressions and equations. So, the dimensions that will produce the maximum area are  $16\frac{2}{3}$  ft by 25 ft.

Again, we need to be very careful to find exactly what the situation is asking for. Sometimes the problem is asking for a value of the vertex, sometimes the problem is asking for the solutions to the quadratic and sometimes the problem is merely asking to evaluate a quadratic function. We must carefully read each question to determine exactly what is being asked.

## 10.6 Exercises

- The number of board feet in a 16 foot long tree is approximated by the model  $F(d) = 0.77d^2 - 1.32d - 9.31$  where  $F$  is the number of feet and  $d$  is the diameter of the log. How many board feet are in a log with diameter 12 inches? What is the diameter that will produce the minimum number of board feet?

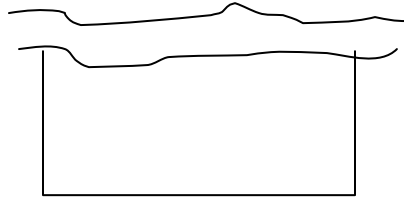
2. The number of horsepower needed to overcome a wind drag on a certain automobile is given by  $N(s) = 0.005s^2 + 0.007s - 0.031$ , where  $s$  is the speed of the car in miles per hour. How much horsepower is needed to overcome the wind drag on this car if it is traveling 50 miles per hour? At what speed will the car need to use 200 horsepower to overcome the wind drag?
3. The number of baseball games that must be scheduled in a league with  $n$  teams is given by  $G(n) = \frac{n^2 - n}{2}$  where each team plays every other team exactly once. A league schedules 15 games. How many teams are in the league?
4. For the years of 1983 to 1990, the number of mountain bike owners  $m$  (in millions) in the US can be approximated by the model  $m = 0.337t^2 - 2.265t + 3.962$ ,  $3 \leq t \leq 10$  where  $t = 3$  represents 1983. In which year did 2.5 million people own mountain bikes? In what year was the number of mountain bike owners at a minimum?
5. A manufacturer of tennis balls has a daily cost of  $C(x) = 200 - 10x + 0.01x^2$  where  $C$  is the total cost in dollars and  $x$  is the number of tennis balls produced. What number of tennis balls will produce the minimum?
6. The value of Jennifer's stock portfolio is given by the function  $v(t) = 50 + 73t - 3t^2$ , where  $v$  is the value of the portfolio in hundreds of dollars and  $t$  is the time in months. How much money did Jennifer start with? When will the value of Jennifer's portfolio be at a maximum?
7. The value of Jon's stock portfolio is given by the function  $v(t) = 50 + 77t + 3t^2$  where  $v$  is the value of the portfolio in hundreds of dollars and  $t$  is the time in months. How much money did Jon start with? What is the minimum value of Jon's portfolio?
8. Find the number of units that produce the maximum revenue  $R = 900x - 0.1x^2$ , where  $R$  is the total revenue (in dollars) and  $x$  is the number of units sold.
9. A textile manufacturer has daily production costs of  $C = 10,000 - 110x + 0.045x^2$ , where  $C$  is the total cost (in dollars) and  $x$  is the number of units produced. How many units should be produced each day to yield a minimum cost?
10. A manufacturer of light fixtures has daily production costs of  $C(x) = 800 - 8x + 0.25x^2$ , where  $C$  is the total cost (in dollars) and  $x$  is the number of units produced. How many units should be produced every day to yield a minimum cost?
11. A company's weekly revenue in dollars is given by  $R(x) = 2000x - 2x^2$ , where  $x$  is the number of items produced during a week. What amount of items will produce the maximum revenue?
12. A company earns a weekly profit of  $P$  dollars by selling  $x$  items, according to the equation  $P(x) = -0.5x^2 + 40x - 300$ . How many items does the company have to sell each week to maximize the profit?

13. Advertising revenue for newspapers in the United States for the years 1985 through 1999 is approximated by the model  $R = -1.03 + 7.11t - 0.38t^2$  where R is revenue in billions of dollars and t represents the year with  $t = 5$  corresponding to the year 1985. In what year will revenue be maximum?
14. A ball rolls down a slope and travels a distance  $d = 6t + \frac{t^2}{2}$  feet in  $t$  seconds. Find when the distance is 17 feet.
15. The height in feet of a bottle rocket is given by  $h(t) = 160t - 16t^2$  where  $t$  is the time in seconds. How long will it take for the rocket to return to the ground? What is the height after 2 seconds?
16. A foul ball leaves the end of a baseball bat and travels according to the formula  $h(t) = 64t - 16t^2$  where  $h$  is the height of the ball in feet and  $t$  is the time in seconds. How long will it take for the ball to reach a height of 64 feet in the air?
17. A model rocket is projected straight upward from the ground level according to the height equation  $h = -16t^2 + 192t$ ,  $t > 0$ , where  $h$  is the height in feet and  $t$  is the time in seconds. At what time is the height of the rocket maximum and what is that height?
18. Emma hits a golf ball of the tee. The height of the ball is given by  $y = -16x^2 + 4013x + 3250$  where  $y$  is the height in yards above the ground and  $x$  is the horizontal distance from the tee in yards. How far does Emma hit the ball? What is the maximum height of the ball?
19. Jon is hitting baseballs. When he tosses the ball into the air, his hand is 5 feet above the ground. He hits the ball when it falls back to a height of 4 feet. The height of the ball is given by  $h = 5 + 25t - 16t^2$ , where  $t$  is in seconds. How much time will pass before Jon hits the ball? What is the maximum height the ball attains?
20. The height  $h$  in feet of a projectile launched vertically upward from the top of a 96-foot tall tower when time  $t = 0$  is given by  $h = 96 + 80t - 16t^2$ . How long will it take the projectile to strike the ground? What is the maximum height that the projectile reaches?
21. The formula  $h = -16t^2 + 48t + 160$  gives the height of an object thrown from a building 160 feet high with an initial speed of 48 ft/sec, where  $t$  is measured in seconds. Find the time for the object to hit the ground and find the maximum height of the object.
22. While playing basketball this weekend Frank shoots an air-ball. The height  $h$  in feet of the ball is given by  $h = -16t^2 + 32t + 8$ . How long will it take the ball to strike the ground? What is the maximum height of the ball?
23. While on an Audubon field trip Jennifer sees a Red-Tail Hawk drop its prey. The height  $h$  in feet of the prey is given by  $h = -16t^2 + 48t + 50$ . How long will it take the prey to strike the ground? What is the maximum height of the prey?

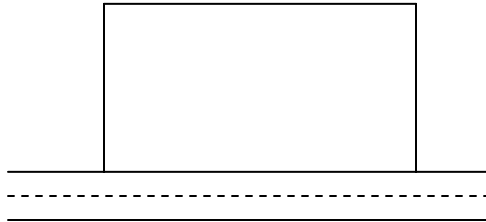
24. While playing catch with his grandson yesterday Tim throws a ball as hard as possible into the air. The height  $h$  in feet of the ball is given by  $h = -16t^2 + 64t + 8$ , where  $t$  is in seconds. How long will it take until the ball reaches the grandson's glove if he catches it at a height of 3 feet? What is the maximum height of the ball?
25. The path of a high diver is given by  $y = -\frac{4}{9}x^2 + \frac{24}{9}x + 10$  where  $y$  is the height in feet above the water and  $x$  is the horizontal distance from the end of the diving board in feet. What is the maximum height of the diver and how far out from the end of the diving board is the diver when he hits the water?
26. The height  $h$  in feet of a projectile launched vertically upward from the top of a 96-foot tall bridge is given by  $h = 90 + 16t - 16t^2$  where  $t$  is time in seconds. What is the maximum height and how long will it take the projectile to strike the ground?
27. The height  $h$  in feet of a projectile launched vertically upward from the top of a 32-foot tall bridge is given by  $h = 38 + 16t - 16t^2$  where  $t$  is time in seconds. When does the projectile reach a maximum height and how long will it take the projectile to strike the ground?
28. The height  $h$  in feet of a projectile launched vertically upward from the top of a 280-foot tall bridge is given by  $h = 280 + 48t - 16t^2$  where  $t$  is time in seconds. When does the projectile reach a maximum height and how long will it take the projectile to strike the ground?
29. The length of a rectangular flower garden is 5 feet more than its width. If the area of the garden is 104 square feet, find the dimensions of the flower garden.
30. The height of a triangular flower garden is 6 feet more than the length of the base. If the area of the garden is 8 square feet, find the dimensions of the flower garden.
31. The length of a rectangular plot of land is 10 yards more than its width. If the area of the land is 600 square yards, find the dimensions of the plot of land.
32. The length of a rectangular plot of land is 10 yards more than its width. If the area of the land is 600 square yards, find the dimensions of the plot of land.
33. The height of a triangular window is 3 feet less than its base. If the area of the window is 20 square feet, find the dimensions of the window.
34. The length of a Ping-Pong table is 3 ft more than twice the width. The area of a Ping-Pong table is 90 square feet. What are the dimensions of a Ping-Pong table?
35. The width of a rectangle is five less than twice the length. What is the minimum area of such a rectangle?
36. The length of a rectangle is one more than the width. What is the minimum area of such a rectangle?
37. The base of a triangle is one more than four times the height. Determine the dimensions that will give a total area of  $9 \text{ cm}^2$ . What is the minimum area of such a triangle?
38. The height of a triangle is two more than three times the base. Determine the dimensions that will give a total area of  $28 \text{ yds}^2$ . What is the minimum area of such a triangle?



39. The perimeter of a rectangle is 50 yds. What are the dimensions that will produce the maximum area of such a rectangle?
40. The perimeter of a rectangle is 70 m. What are the dimensions that will produce the maximum area of such a rectangle?
41. The perimeter of a rectangle is 100 ft. What is the maximum area of such a rectangle?
42. The perimeter of a rectangle is 120 cm. What is the maximum area of such a rectangle?
43. Three hundred feet of fencing is available to enclose a rectangular yard along side of the St. John's River, which is one side of the rectangle as seen below. What dimensions will produce an area of  $10,000 \text{ ft}^2$ ? What is the maximum area that can be enclosed?



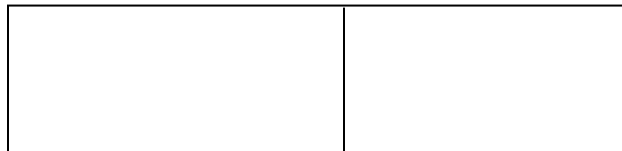
44. Five hundred feet of fencing is available to enclose a rectangular lot along side of highway 65. Cal Trans will supply the fencing for the side along the highway, so only three sides are needed as seen below. What dimensions will produce an area of  $40,000 \text{ ft}^2$ ? What is the maximum area that can be enclosed?



45. Two rectangular pens are to be made from 200 yds of fencing as seen below. Determine the dimensions that will produce the maximum area.



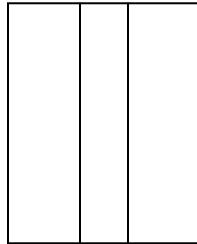
46. Two rectangular lots are to be made from 400 ft of fencing as seen below. Determine the dimensions that will produce the maximum area.



47. Three rectangular corrals are to be made from 800 meters of fencing as seen below. Determine the dimensions that will produce the maximum area. What is area of one of the corrals?



48. Three rectangular corrals are to be made from 100 yards of fencing as seen below. Determine the dimensions that will produce the maximum area. What is area of one of the corrals?



49. John V. wants to fence three sides of a rectangular exercise yard for his dog. The fourth side of the exercise yard will be a side of the house. He has 80 feet of fencing available. Find the dimensions of the exercise yard that will produce the maximum enclosed area.
50. Charles wants to build a vegetable garden such that three sides of the garden are fenced and the fourth side of the garden will be the existing back fence. He has 30 feet of fencing available. Find the dimensions of the garden that will produce the maximum enclosed area.
51. Chris wants to make an enclosed rectangular area for a mulch pile. She wants to make the enclosure in such a way as to use a corner of her back yard. She also wants it to be twice as long as it is wide. Since the yard is already fenced, she simply needs to construct two sides of the mulch pile enclosure. She has only 15 feet of material available. Find the dimensions of the enclosure that will produce the maximum area.
52. Paul, a rancher, has 200 ft. of fencing to enclose two adjacent rectangular corrals. What are the dimensions that will produce the maximum enclosed area?
53. Patrick has 1400 ft. of irrigation piping he wants to use to irrigate his back lawn. He wants to lay the piping in such a manor as to cut off three equal sized rectangular regions in the yard. What are the dimensions that will produce the maximum enclosed area?
54. Show that among all rectangles of fixed perimeter  $p$  the one with the largest area is a square.
55. An Athletic field with a perimeter of  $\frac{1}{4}$  mile consists of a rectangle with a semicircle at each end, as shown below. Find the dimensions that yield the greatest possible area for the rectangular region.

