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Gauss' Method for Summing Consecutive Numbers Teacher's Guide

Background: There are a lot of stories that have come up over the centuries about famous mathematicians. One of my favorites is the story about the early life of Karl Freidrick Gauss. The version of the story I know goes like this: One day when Gauss was just a boy, probably no older than 10, his teacher came down with a migraine. In order to get some quiet time, the instructor told the class to add up all the numbers from one to one hundred, figuring this would take them an hour at least. Ten minutes later young Gauss informed the instructor that he was done. Not only that, his answer was correct, the sum of all the whole numbers from one to one hundred is 5050.

Like all great mathematicians Gauss was not only tremendously gifted, but fundamentally lazy. Rather than do the rote work assigned he took the time to find some clever way out of doing any hard work. He noticed that 100 + 1 = 101, 99 + 2 = 101, 98 + 3 = 101, and so on. Finally one gets 50 + 51 = 101. Gauss' then realized that adding all the whole numbers from one to one hundred gives the same sum as adding fifty 101's together. But this is the same amount given by 50 * 101 = 5050.

This shortcut to getting the sum of all the numbers from one to one hundred generalizes to any sum of consecutive whole numbers. Try adding up all the numbers between 1 and 23 by copying Gauss' method before reading on.

The natural place to start is by checking that 23 + 1 = 24, 22 + 2 = 24, and so on. Now you just need how many 24's you can make like this. You probably ran into a glitch when you realized that 12 doesn't get "paired off" with anything. There are a couple of ways around this. You could just compute 24 * 11 and then add 12. Or, you could use Gauss' method to add all the numbers between one and twenty-two, then add 23. If you use this latter method you see that the sum of all the numbers from one to twenty three is (23*11) + 23, or 23 * 12 = 276.

There is no reason you can't use this method when you start from a number other than one either. Try using Gauss' method to find the sum of all the numbers from 13 to 39 before reading on.

Notice that 13 + 39 = 52, 14 + 38 = 52, 15 + 37 = 52, and so on. If you keep going, you find that the sum is equal to the sum of thirteen copies of 52, or 52 * 13 = 677. Here is a statement of the general rule:

Let x and y be positive, whole numbers. The sum of all the whole numbers from x to y is equal to
$(x + y) * [(x - y)/2]$ if $x - y$ is an odd number,
$(x + y - 1) * \{[(x-y)/2] + 1\}$ if $x - y$ is an even number.

If you don't understand why these formulas always work, do some more examples and it should start to become clear. This is the most important part of your preparation. If you don't understand why this method works well enough to explain it to someone your own age, you won't be able to deal with student's questions that will come up during the activity

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Teaching Gauss' Method: An activity for learning Gauss' method is appropriate for any age group of students. The prerequisites they need are the ability to add numbers with many digits, and they should at least know what multiplication is even they are not fluent with their multiplication tables. It will be difficult to run this activity unless the students have some experience with multi-digit multiplication. However, if you feel it is appropriate for your classroom you can have the students use calculators and get around this.

This lesson offers an excellent opportunity for students to practice their skills at adding and multiplying numbers with many digits. The fact that there work is heading toward a creative end will help motivate this practice. It also allows students to participate in developing an advanced procedure for adding easily computing complicated sums, and helps to illustrate the relationship between the operations of addition and multiplication.

For this activity you should split the class into small groups, six to a group at the very most and four or five to a group works better. Each group will need something to record their work on (like a sheet of butcher paper) and every student should have a writing utensil. You can use whatever visual aid you like to demonstrate calculations (overhead, chalkboard, etc.). You will also need a large number of some small objects and a container to put them in. Marbles and a jar work well, tiles and a box, or even sheets of paper in a drawer are examples. Use whatever you have handy.

Start at a convenient spot in your classroom and have a student come put one object in the container. Choose another student and have them put two objects in the container. The third student should then put three objects in the container, etc. Keep going until every student in the classroom as placed objects in the container. It is easiest to do the first problem with an even number of summands, so if you have an odd number of students, place one last set of objects in the container yourself after all of the students have gone.

Ask the class if they know a way to find out how many objects are in the container. It may take a little prodding and some discussion but eventually the students should come up with the idea to sum 1,2,3, and so on up to the number of students in the class. Have the students work in small groups to find the sum. Tour the room to keep an eye on their progress and make sure each group is on task. Make sure they have a way to record their process, like a large sheet of paper for each group to do calculations on. If your class commonly uses manipulatives or other resources with math problems, grant access to these as well.

This is the part of the exercise that lets the students practice their strategies for addition. The students may come up with some innovative strategies for adding so many summands together. Rearranging some of the numbers to produce multiples of ten is a common approach. Some groups may simply try to use "brute force" and do the addition as "1 + 2 = 3, 3 + 3 = 6, 6 + 4 = 10," and so on. This is OK. Some groups might even figure out Gauss' method themselves, this is OK as well! Above all, prepare for the students to take some unusual approaches and be ready for questions you think might arise.

After most of the groups have either finished or are close to doing so, have each group present their answer and how they got to it. Chances are at least some of the groups tried rearranging the summands to make "nice" numbers to add up. Write out the sum so that the entire class can see it (on the board, on an overhead projector, etc.), and ask the students to work in their groups to try to rearrange the numbers to make them easier to add.

Don't give the students quite as much time for this as the last task, but go around and monitor their progress. When you get to a point where you can have a discussion as a whole class, have the students stop and let each group explain how they tried to rearrange the numbers. Have a class discussion about what is good or bad about each method and whether the students think there is an easier way. You'll be surprised at how nice some of the students' ideas are!

If any of the groups actually came up with Gauss' method, then after each group has presented and you have come to a stopping point in the class discussion you can try to lead the class discussion into turning the addition problem into one of multiplication. If none of the groups came up with the method on their own, point out how the first and last numbers add to the same amount as the second and second to last numbers and so on, and ask the students to work in groups to use this pattern to figure out the sum. Then ask the class if the problem can be turned into one of multiplication and discuss how this will get the correct answer.

At this point have each group tackle another sum using the method you demonstrated. Then have the class come together and check each answer. Now the students are ready to work on their own. If you have time, you can assign the worksheet as class work and monitor the students' progress, or assign it for homework.

Either way, the next time the class meets review the worksheet as a group. Discuss as a class whether or not the methods shown in the lesson and the worksheet will work for any sum, and for what kinds. The last portion in particular may be very challenging. If any of the students came up with the method given earlier for adding an odd number of summands, have them demonstrate and discuss as a class how to use the method for any consecutive sum of an odd number of summands.

If none of the students found the solution presented here, have them break into groups again. Demonstrate that the problem can be done by summing all the numbers from 1 to 22, then adding 23 to that sum. After each group works out the solution using this method, bring the class back together and discuss how to add an odd number of consecutive summands.

After the class has arrived at Gauss' solutions for summing consecutive integers, be sure to summarize what they learned in this lesson. The problem of adding up consecutive numbers can be more easily solved by turning it into a multiplication problem. The way that the multiplication works is different if there are an even or odd number of summands. It doesn't matter what number you start at, as long as the numbers are consecutive. You may also want the students to put down the method in their notes for future reference.

Student Worksheet **Name**

1. Find the sum: 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 + 31 + 32 + 33 + 34 + 35. Show all your work and explain why you solved the problem the way you did.

2. **A.** What is 22 + 35? 23 + 34? 24 + 33?
   B. Find the sum from **1** using multiplication. Why is your answer correct?

3. **A.** Find the sum of all the numbers from 1 to 33. Show all of your work and explain why you solved the problem the way you did.

   B. How was this problem different from the other problems you did in class and question **1**? What did you do differently to answer this problem?